

Polyptych Varieties

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joint with

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Toric Varieties

$$N = \mathbb{Z}^r$$

$$M = (\mathbb{Z}^r)^* = \text{Hom}(N, \mathbb{Z})$$

$$N \times M \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{Z}$$

$$T = N \otimes \mathbb{C}^* \\ \cong (\mathbb{C}^*)^r$$

$$\text{Hom}(T, \mathbb{C}^*) \cong M$$

$$\text{Hom}(\mathbb{C}^*, T) \cong N$$

Toric Varieties

characters

$$u \in M \iff t^u: T \rightarrow \mathbb{C}^* \in \mathbb{C}$$

$$\mathbb{C}[T] = \mathbb{C}[t^u \mid u \in M]$$

$$t^{u_1} t^{u_2} = t^{u_1 + u_2}$$

cocharacters

$$v \in N \iff v: \mathbb{C}^* \rightarrow T$$

valuations

$$\text{val}_v: \mathbb{C}[T] \rightarrow \overline{\mathbb{Z}} = \mathbb{Z} \cup \{\infty\}$$

$$\text{val}_v(t^u) = \langle v, u \rangle$$

$$\text{val}_v(\sum c_u t^u) = \min \{ \langle v, u \rangle \mid c_u \neq 0 \}$$

Toric Varieties

half spaces

$$H_{v, \geq 0} = \{u \mid \langle v, u \rangle \geq 0\} \subseteq M$$

divisors

$$\operatorname{Spec}(\mathbb{C}[t^u \mid u \in H_{v, \geq 0}]) = X_v \supset T$$

$$X_v \setminus T = D_v \cong T / v \mathbb{C}^*$$

$$\operatorname{ord}_{D_v}(f) = \operatorname{val}_v(f)$$

Toric Varieties

Fans

Fix a set of primitive $v_1, \dots, v_n \in N$

For $\alpha \in \{v_1, \dots, v_n\}$

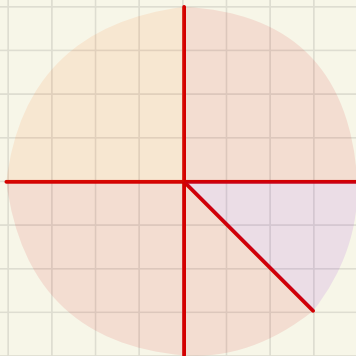
$$\sigma_\alpha = \mathbb{R}_{\geq 0} \{v_i \in \alpha\}$$

A fan Σ on $\{v_1, \dots, v_n\}$

is a set of σ_α such that

$\sigma_\alpha \cap \sigma_\beta$ is a face of $\sigma_\alpha, \sigma_\beta$

Any face of $\sigma_\alpha \in \Sigma$ is in Σ



Toric Varieties

fans $\leftrightarrow$ T -equivariant normal
partial compactifications
of T

$$\Sigma \leftrightarrow X_\Sigma$$

$$\{v_1, \dots, v_n\} \leftrightarrow X_\Sigma \setminus T = D_{v_1} \cup \dots \cup D_{v_n}$$

$$\text{ord}_{D_{v_i}}(f) = \text{val}_{v_i}(f)$$

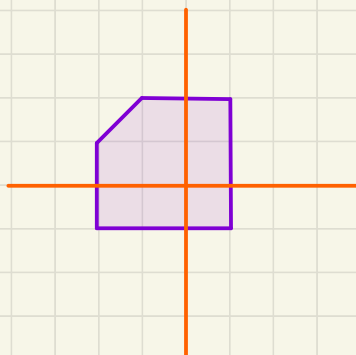
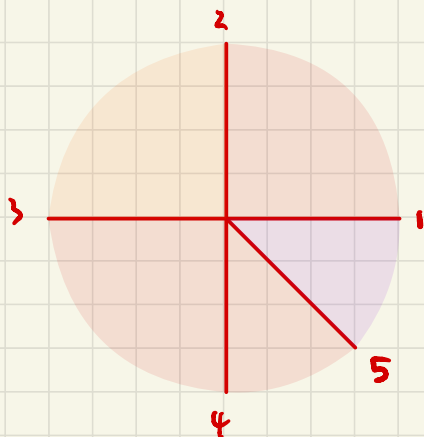
Toric Varieties

Polyhedra in $M \otimes \mathbb{R} \longleftrightarrow$ spaces of global sections

$$\Gamma(X_\Sigma, \mathcal{O}(a_1 D_{\sigma_1} + \dots + a_n D_{\sigma_n}))$$

$$= \mathbb{C} \{ t^u \mid \langle v_i, u \rangle \geq -a_i \}$$

$$= \{ t^u \mid u \in \bigcap H_{v_i, -a_i} \}$$



$$2D_1 + D_2 + D_3 + 2D_4 + 3D_5$$

Toric Varieties

X_Σ proper : Σ complete

smooth : Σ "smooth"

projective : Σ inner normal fan
of a polytope

- class group

- Picard group

convexity in $M \otimes \mathbb{R} \iff$ regular functions
global sections
on X_Σ

Beyond Toric Varieties

- flag varieties, spherical varieties
coloured fans, crystal bases
- T varieties $\dim T < \dim X$
polyhedral divisors, divisorial fans
- Toric vector bundles
Klyachko data, Parliaments of Polytopes
- Mori Dream Spaces
Bunched rings

Generalizing Toric Combinatorics

Toric Degeneration (M-side)

$$\begin{array}{c}
 (E, \tilde{\mathcal{L}}) \\
 \downarrow \pi \\
 \mathbb{A}^1
 \end{array}
 \quad
 \begin{array}{l}
 \tilde{\mathcal{L}} \text{ relatively ample} \\
 \text{flat, } \mathbb{G}_m\text{-equivariant}
 \end{array}$$

$$\pi^{-1}(1) = (X, \mathcal{L}) \quad \pi^{-1}(0) = (X_{\Delta}, \mathcal{L}_{\Delta})$$

Toric

$$(X, \mathcal{L}) \Rightarrow (X_{\Delta}, \mathcal{L}_{\Delta})$$

Generalizing Toric Combinatorics

Toric Degeneration (M-side)

Karch, M:

$$(X, \mathcal{L}) \Rightarrow (X_\Delta, \mathcal{L}_\Delta)$$

corresponds to

$$\sigma \in \text{Trop}(X)$$

Generalizing Toric Combinatorics

Toric Degeneration (M-side)

Escobar, Harada:

Let $(X, \mathcal{L}) \Rightarrow (X_{\Delta_1}, \mathcal{L}_{\Delta_1}), (X_{\Delta_2}, \mathcal{L}_{\Delta_2})$

$$\Delta_1 \in M_1 \otimes \mathbb{R}, \Delta_2 \in M_2 \otimes \mathbb{R}$$

correspond to $\sigma_1, \sigma_2 \in \text{Trop}(X)$

with $\sigma_1 \cap \sigma_2$ a facet of σ_1, σ_2 .

Then there is a piecewise linear map

$$\mu_{21} : M_1 \otimes \mathbb{R} \rightarrow M_2 \otimes \mathbb{R}$$

$$\text{with } \mu_{21}(\Delta_1) = \Delta_2$$

Piecewise linear maps between polytopes

- Petrucci: Moduli of Fano's
- Ilten: Complexity 1 T-varieties

Generalizing Toric Combinatorics

Tropical Skeleton (N -side)

- Gross, Hacking, Keel

$$\bigcup_{i \in \mathcal{I}} T_i \subseteq \mathcal{U}$$

open

$$T_i = N_i \otimes \mathbb{C}^*$$

The birational equivalences $T_i \cong T_j$

induce piecewise linear bijections

$$\mu_{ij} : N_i \rightarrow N_j$$

Cluster Algebras merge these stories

- Rietsch, Williams

- Bossinger, Cheung, Magee, Nájera-Chávez

Polyptych Lattices

$$\mathcal{M} = \{M_i, \mu_{ij} \mid i \in \mathcal{I}\}$$

$$M_i \simeq \mathbb{Z}^r$$

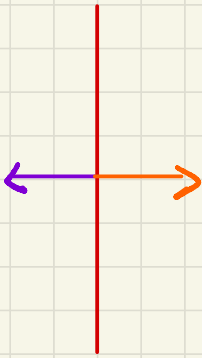
$$\mu_{ij}: M_j \longrightarrow M_i \quad \text{piecewise linear bijection}$$

$$\mu_{ij} \circ \mu_{jk} = \mu_{ik} \quad \text{cocycle condition}$$

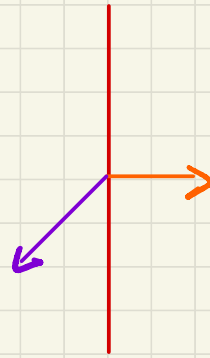
Example

$$\mu_{12}(a, b) = (a, b - \min\{a, 0\})$$

$$\mu_{21}(x, y) = (x, y + \min\{x, 0\})$$



M_1



M_2

Polyptych Lattices

Elements

$$\mathcal{M} = \bigcup_{i \in \mathcal{I}} M_i / \mu_i; (m) \sim m$$

$$\mathcal{M} \otimes \mathbb{R} \text{ real elements} \quad (M_i \otimes \mathbb{R})$$

Chart maps

$$\phi_i: \mathcal{M} \rightarrow M_i$$

Chart Addition

$$m_1 +_i m_2 = \phi_i^{-1}(\phi_i m_1 + \phi_i m_2)$$

Convexity

"chart-wise convexity"

$C \subseteq \mathcal{M} \otimes \mathbb{R}$ is chartwise convex
if $\phi_i|_C$ is convex $\forall i \in I$

Lines

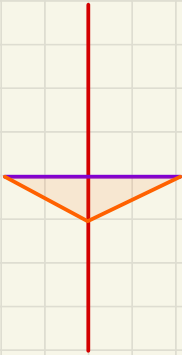
$l \subseteq \mathcal{M} \otimes \mathbb{R}$ is a line segment if $\phi_i|_l$ is a straight line segment for some chart M_i

C chartwise convex

$\longleftrightarrow$

$u, v \in C \quad u \xrightarrow{l} v \Rightarrow l \subseteq C$

Example



m_1



m_2



Points

$$\rho: \mathcal{M} \rightarrow \mathbb{Z}$$

$$\rho(m_1) + \rho(m_2) = \min \{ \rho(m_1 + i m_2) \mid i \in \mathbb{I} \}$$

$\Pi(\mathcal{M}) = \text{set of points}$

For any $\rho \in \Pi(\mathcal{M})$, $\phi_i: \mathcal{M} \rightarrow M_i$

$\rho \phi_i^{-1}$ is piecewise linear, convex on M_i .

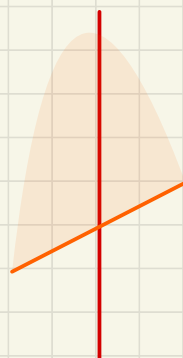
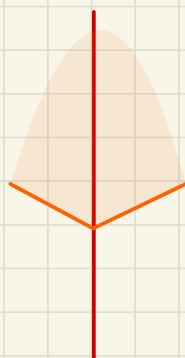
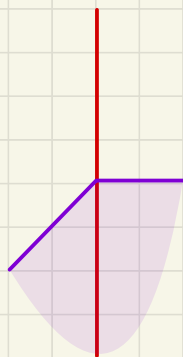
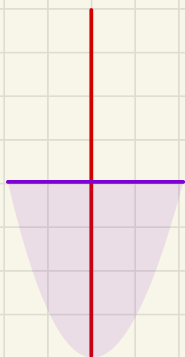
Pointwise Convexity

$H_{p, \geq a} = \{m \mid p(m) \geq a\} \subseteq \mathcal{M} \otimes \mathbb{R}$
is chartwise convex.

- $p_{\text{conv}}(C) = \bigcap_{C \subseteq H_{p, a}} H_{p, a}$

- C is "pointwise convex" if $C = p_{\text{conv}}(C)$

Example



M_1

M_2

Canonical Semialgebra

S_M is the semigroup

$$= \left\{ [m] \mid m \in M \quad \bigoplus_{i \in X} [m_i] = \bigoplus_{j \in Y} [n_j] \right. \\ \left. \text{if } \text{pconv}(X) = \text{pconv}(Y) \right\}$$

with product

$$[m_1] \star [m_2] = \bigoplus_{i \in I} [m_1 +_i m_2]$$

$\star$ is commutative and associative

S_M is an idempotent semiring

Dual Polytopal Lattices

$\mathcal{P}_M$ - set of functions $\phi: M \rightarrow \mathbb{Z}$
such that $\phi_i^{-1}\psi: M_i \rightarrow \mathbb{Z}$
is piecewise-linear, convex.

M, N are dual if

$$S_M \cong \mathcal{P}_N \quad M \leftrightarrow \Pi(N)$$

$$S_N \cong \mathcal{P}_M \quad N \leftrightarrow \Pi(M)$$

Inducing a pairing

$$N \times M \rightarrow \mathbb{Z}$$

$$n, m \mapsto \langle n, m \rangle$$

Dual Polytych Lattices

$\mathcal{M}$ is finite type iff $|\mathcal{I}| < \infty$.

For $\mathcal{M}$ finite type let $\Sigma \mathcal{M}$

denote the common refinement of

the subdivisions defined by μ_{ij}

into domains of linearity.

Thm: If $\mathcal{M}$ is finite type

then $\mathcal{M}$ has a dual $\mathcal{N}$ iff

① Each μ_{ij} is convex.

② Each $\gamma \in \Sigma \mathcal{M}$ is pointwise convex.

$\mathcal{N}$ is finite and unique
up to isomorphism.

Dual Polytych Lattices

Example (Escobar, Fris-Medina, Hadravský, Magee, M.)

Let A be a finite type
cluster algebra with $U = \text{Spec}(A)$

$\text{Trop}(U)$ is a self-dual polytych lattice

Dual Polytych Lattices

Example

$$M_{r,d} \quad r, d \geq 1$$

$$M_{r,d}(i) = \mathbb{Z}^{r+d-1} = \{(a_1, \dots, a_r, b_1, \dots, \hat{b}_i, \dots, b_d)\}$$

$$(\bar{a}, b_1, \dots, \hat{b}_i, \dots, b_d)$$

$$\downarrow \mu_{j,i}$$

$$(\bar{a}, b_1, \dots, b_{i-1}, \min(\bar{a}) - \sum b_j, b_{i+1}, \dots, \hat{b}_i, \dots, b_d)$$

$M_{r,d}$ and $M_{d,r}$ are dual.

$$x_1, \dots, x_r - y_1 - \dots - y_d$$

Geometry and Detropicalization.

$\mathcal{M}$ = a finite type Polytopal lattice

$\mathcal{N}$ = dual of $\mathcal{M}$

Combinatorics

$\mathcal{N}$ - lattice

$\mathcal{N}$ - polytopal
lattice

Geometry

$T_{\mathcal{N}}$

?

Geometry and Detropicalization.

A/c

$v: A \rightarrow S_m$ is a valuation if

$$(1) \quad v(fg) = v(f) \otimes v(g)$$

$$(2) \quad v(f+g) \geq v(f) \oplus v(g)$$

$$(3) \quad v(cf) = v(f) \quad c \in \mathbb{C}^*$$

$$(4) \quad v(0) = \infty$$

A is a Detropicalization of m

if there is a valuation

$$v: A \rightarrow S_m$$

such that for every $[m] \in S_m$

there is a $f_m \in A$ with $v(f_m) = [m]$.

Geometry and Detropicalization.

Example

If $\mathcal{M}$ has one chart, $\approx \mathbb{R}^r$

$S_{\mathcal{M}} =$ convex piecewise linear functions of $(\mathbb{R}^r)^*$

$$\sum C_{\alpha} t^{\alpha} \longmapsto \text{MIN} \{ C_{\alpha} \neq 0 \mid \langle -, \alpha \rangle \}$$

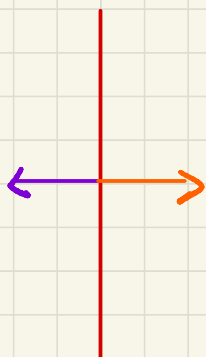
Makes $\mathbb{C}[T]$ a detropicalization of $\mathbb{R}^r$

Example

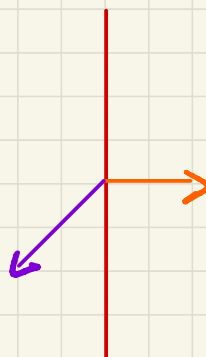
$$\hookrightarrow [x, y, t^{\pm}] / xy - t - 1$$

$$\mu_{12}(a, b) = (a, \min\{a, 0\} - b)$$

$$\mu_{21}(x, y) = (x, y + \min\{x, 0\})$$



M_1



M_2

Geometry and Detropicalization

$\mathcal{M}, \mathcal{N}$ finite, dual.

Let A be a detropicalization of $\mathcal{M}$.

Let $\mathcal{U} = \text{Spec}(A)$.

-) For every chart M_i There is
a flat degeneration

$$\mathcal{U} \Rightarrow T^{M_i} = \text{Hom}(M_i, \mathbb{C}^*)$$

-) For every chart N_j There is an
open torus $N_j \otimes \mathbb{C}^* = T_{N_j} \hookrightarrow \mathcal{U}$

Geometry and Detropicalization

$\mathcal{M}, \mathcal{N}$ finite, dual.

Let A be a detropicalization of $\mathcal{M}$.

Let $\mathcal{U} = \text{Spec}(A)$.

→ The subset $\{s_m \mid m \in \mathcal{M}\} \subseteq A$ is a basis

→ There is a finite generating set $B \subseteq A$.
 A is normal.

→ Each $n \in \mathcal{N}$ defines an
integral valuation $\text{val}_n: A \rightarrow \mathbb{Z}$

Geometry and Detropicalization

$\mathcal{M}, \mathcal{N}$ finite, dual

A a detropicalization of $\mathcal{M}$

$\Delta \in \mathcal{M}$ is a **polytope** if $\phi_i \Delta$ is a polytope for all $i \in \mathbb{I}$.

If $0 \in \Delta^\circ$ then

- Δ is chart convex and point convex
- There is a projective compactification $\mathcal{U} \subseteq X_\Delta$
- $X_\Delta \setminus \mathcal{U} = D_1 \cup \dots \cup D_\ell$ where $\text{ord}_{D_i} = \sum a_i n_i$, $\Delta = \bigcap_{i=1}^{\ell} H_{n_i, a_i}$

Geometry and Detropicalization

$\mathcal{M}, \mathcal{N}$ finite, dual, A a detropicalization of $\mathcal{M}$

β - the distinguished generating set of A

Let $\mathcal{P} = \{\alpha_1, \dots, \alpha_k\} \subseteq \mathcal{N}$

For every $\alpha \in \mathcal{P}, \beta \in \beta$ there is

a subset $\sigma_{\alpha, \beta} \subseteq \mathcal{N}$ (finite union of polyhedra)

"weak face" associated to α, β .

A collection $\{\sigma_{\alpha, \beta}\} = \Sigma$ is

a "weak fan" if:

① $\sigma_{\alpha, \beta} \in \Sigma \quad \alpha \leq \alpha', \beta \leq \beta' \text{ then } \sigma_{\alpha', \beta'} \in \Sigma$

② $\sigma_{\alpha, \beta} \cap \sigma_{\alpha', \beta'} = \sigma_{\alpha'', \beta''} \in \Sigma$

Geometry and Detropicalization

$\mathcal{M}, \mathcal{N}$ finite, dual, A a detropicalization of $\mathcal{M}$
 A is a UFD.

β - the distinguished generating set of A

$P = \{n_1, \dots, n_2\}$, $\sum$ a fan on β , P

- There is a normal scheme $X_\Sigma \supseteq \mathcal{U}$

$$X_\Sigma \setminus \mathcal{U} = D_1 \cup \dots \cup D_e$$

with $\text{ord}_{D_i} = \text{val}_{n_i}$

- Each weak face $\sigma_{\alpha, \beta} \in \Sigma$
determines an affine open $U_{\alpha, \beta} \subseteq X_\Sigma$

Geometry and Detropicalization

$\mathcal{M}, \mathcal{N}$ finite, dual, A a detropicalization of $\mathcal{M}$
 A is a UFD.

β - the distinguished generating set of A

$P = \{n_1, \dots, n_\ell\}$, $\sum$ a fan on β, P

- X_Σ is proper $\Leftrightarrow \bigcup_{\sigma \in \Sigma} \sigma_{\alpha, \beta} = \mathcal{N}$
 Σ is complete

- Global sections of divisors $\leftrightarrow$ Polytopes

$$\Gamma(a_1 D_1 + \dots + a_\ell D_\ell) = \mathbb{Q}\{f_m \mid m \in \Delta\}$$

$$\Delta = \bigcap_{i=1}^r H_{n_i, -a_i}$$

Geometry and Detropicalization

$\mathcal{M}, \mathcal{N}$ finite, dual, A a detropicalization of $\mathcal{M}$
 A is a UFD.

β - the distinguished generating set of A

$P = \{n_1, \dots, n_\ell\}$, Σ a fan on β , P

- A polytope $\Delta \in \mathcal{M}$ defines a dual fan Σ_Δ ,
 X_Σ is projective iff $\Sigma = \Sigma_\Delta$ for some Δ

- In this case there is a toric degeneration

$$(X_\Sigma, \mathcal{L}_\Delta) \Rightarrow (X_{\phi_i \Delta}, \mathcal{L}_{\phi_i \Delta}) \text{ for each } i \in I$$

- $\text{Pic}(X_\Sigma)$, $\text{Pic}(X_\Sigma)$, $\text{NEF}(X_\Sigma)$, $\text{EFF}(X)$

smoothness can all be computed from $\Sigma, \mathcal{U}$

Geometry and Detropicalization

$\mathcal{M}, \mathcal{N}$ finite, dual, A a detropicalization of $\mathcal{M}$
 A is a UFD.

β - the distinguished generating set of A

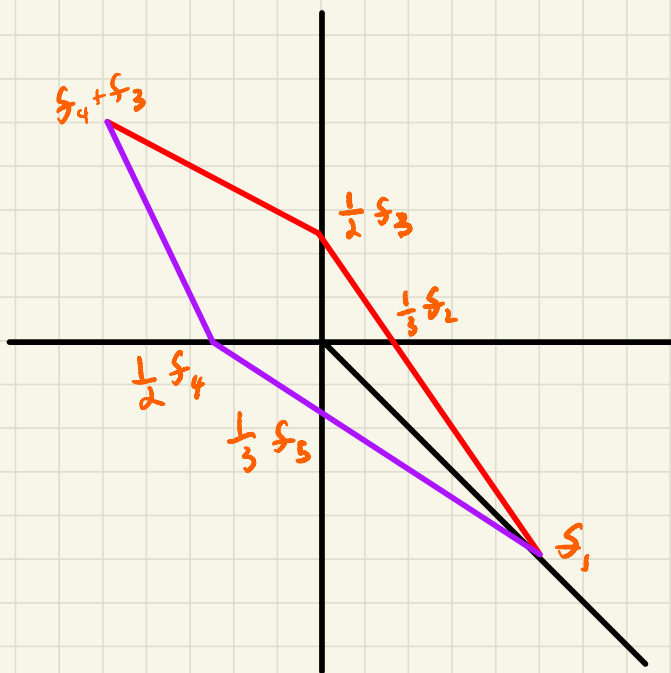
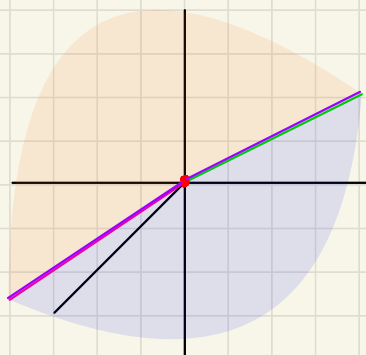
$P = \{n_1, \dots, n_\ell\}$, Σ a fan on β , P

Let $X \supseteq \mathcal{U}$ with $X \setminus \mathcal{U} = D_1 \cup \dots \cup D_\ell$
 $\text{ord}_{D_i} = \forall a \mid n_i$

Then $X = X_\Sigma$ for some fan Σ .

Example

$$\mathbb{C}[f_1, f_2, f_3, f_4, f_5] / \begin{array}{l} f_1 f_3 - f_2 - 1 \\ f_2 f_4 - f_3 - 1 \\ f_3 f_5 - f_4 - 1 \\ f_4 f_1 - f_5 - 1 \\ f_5 f_2 - f_1 - 1 \end{array}$$



$$\mathcal{CL} \simeq \mathbb{Z} \oplus \mathbb{Z}$$

$$\text{Pic} \simeq \mathbb{Z}(-7, 8) + \mathbb{Z}(8, -7)$$

Cox Ring!

$$\mathbb{C}[\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, t_1, t_2]$$

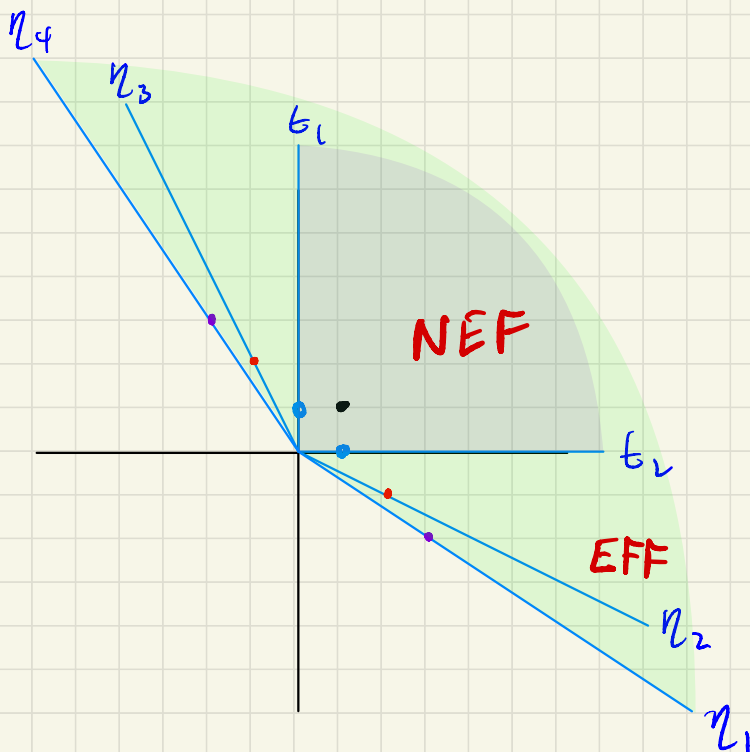
$$\eta_1 \eta_4 - \eta_0 - t_1 t_2$$

$$\eta_0 \eta_2 - \eta_1 t_2^2 - t_1^3$$

$$\eta_1 \eta_3 - t_2 \eta_2 - t_1^2$$

$$\eta_2 \eta_4 - t_1 \eta_3 - t_2^2$$

$$\eta_3 \eta_0 - t_1^2 \eta_4 - t_2^3$$



$$\mathcal{U} \subseteq X$$

- What is The dual?
- What is The fan?

- Toric Ambient Modification

Let $Bl X_\Sigma$ be a blow up at
 $D_{i_1} \cap \dots \cap D_{i_k} \subseteq \partial X_\Sigma$

Then the components of the exceptional
can be read off $n_{i_1} \star \dots \star n_{i_k}$

- Cox Ring of X_Σ

- finitely generated,

polytych $\mathcal{M} \times \mathbb{Z}^l$ ring

Geometry and Detropicalization

$\mathcal{M}, \mathcal{N}$ finite, dual

$A_{\mathcal{M}}$ a detropicalization of $\mathcal{M}$, $\mathcal{U}_{\mathcal{M}} = \text{Spec}(A_{\mathcal{M}})$

$A_{\mathcal{N}}$ a detropicalization of $\mathcal{N}$, $\mathcal{U}_{\mathcal{N}} = \text{Spec}(A_{\mathcal{N}})$

Rietsch-Williams pair:

For every $f \in A_{\mathcal{N}}$ there is a

Polyhedron $\Delta_f \in \mathcal{M}$

and a divisorial compactification $X_f \supseteq \mathcal{U}$

also vice-versa